Symmetric Box-Splines on Root Lattices

BIRS Sampling and Reconstruction: Applications and Advances

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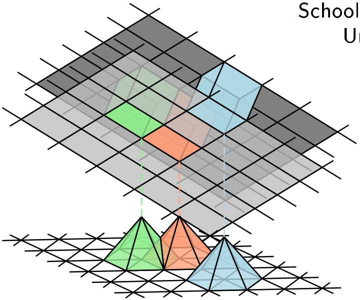
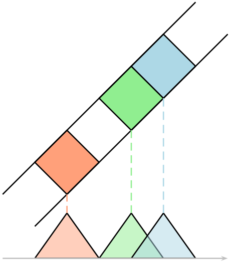
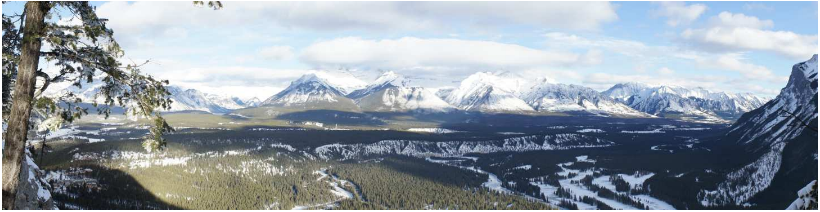






Banff rocks!

Overview

- Cowork with Jörg Peters & Alireza Entezari
- Based on the *sphere packing problem* and *sphere covering problem*, **root lattices** are proposed as efficient sampling lattices in arbitrary dimensions.
- Symmetric **box-spline** filters are constructed for n-dimensional irreducible root lattices, leveraging the symmetric structure of each lattice. ($\mathbb{Z}^n$, $\mathcal{A}_n$, $\mathcal{A}_n^*$, $\mathcal{D}_n$, $\mathcal{D}_n^*$)
- Detailed properties of each box-spline and its spline space are investigated.
- Applications in volume reconstruction are presented.

Minho Kim and Jörg Peters, *Symmetric Box-Splines on Root Lattices*, Journal of Computational and Applied Mathematics (accepted)

Densest Sphere Packing Problem

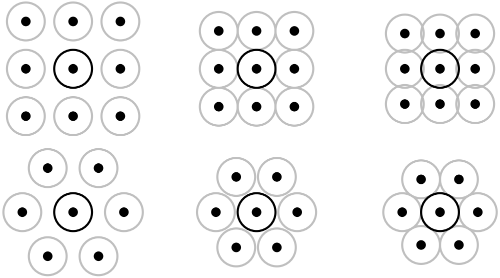
"How can we arrange non-overlapping identical spheres in the n-dimensional Euclidean space maximizing the volume proportion occupied by the spheres?"

- Regular(lattice)/irregular arrangement
- Densest regular packings are known up to dimension 8.

(Courtesy of `mathscareers.org.uk`)

(Courtesy of `old-picture.com`)

Densest Regular Packing and Optimal Sampling Lattices



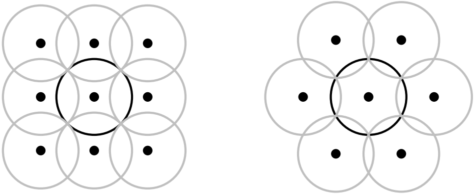
- To find the lattice with maximum *density* Δ:

$$\Delta = \text{proportion of the space occupied by the spheres}$$
$$= \frac{\text{volume of the inscribed sphere}}{\text{volume of the Voronoi cell}}$$

- We want the *inradius* of the Voronoi cells as *large* as possible.

The optimal sampling lattice is the dual of the densest sphere packing lattice (Peteresen and Middleton '62).

Thinnest Sphere Covering



- To find the lattice with minimum thickness Θ:

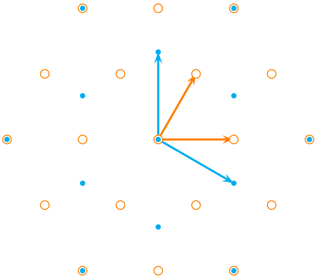
$$\Theta = \text{average } \# \text{ of spheres that contain a point in the space}$$
$$= \frac{\text{volume of the circumsphere}}{\text{volume of the Voronoi cell}}$$

- We want the *circumradius* of the Voronoi cells as *small* as possible.

- For dense regular packing or thin regular covering, high symmetry at every lattice point is required.
  $\rightarrow$ Root lattices

Lattices

- Discrete subgroup of maximal rank in a Euclidean vector space.
- Can be generated by a *square generator matrix* $\mathbf{L}$.
- Dual lattice can be generated by $\mathbf{L}^{-t}$.

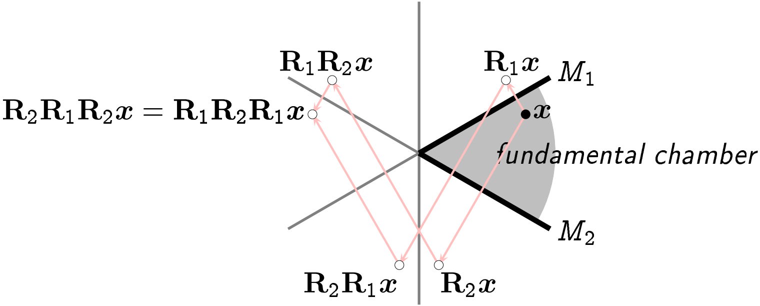


$$\mathbf{L} = \begin{bmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{bmatrix}$$

$$\mathbf{L}^{-t} = \begin{bmatrix} 1 & 0 \\ -1/\sqrt{3} & 2/\sqrt{3} \end{bmatrix}$$

2-Dimensional Example of Finite Reflection Group

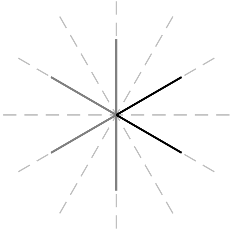
- Why 'reflections'?
  → Reflections generate all the rigid transformations.



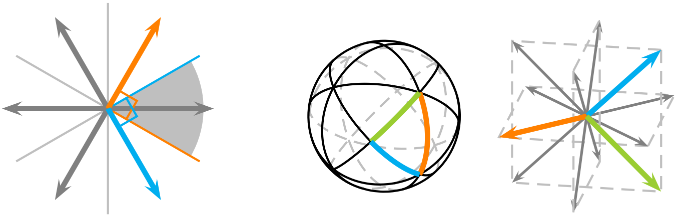
Finite Reflection Groups

"Which configurations of mirrors result in finite reflection groups in n-dimensional Euclidean space?"

- Answered by H.S.M. Coxeter (1907–2003).
- For $n = 2$, dihedral angles π/k, $k \geq 2$ and $k \in \mathbb{Z}$, are allowed.
- For $n > 2$, only finitely many finite reflection groups exist.
- Symmetric: Invariant under the orthogonal transformations generated by reflections.



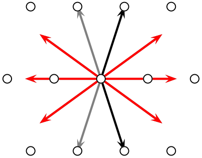
Root Systems



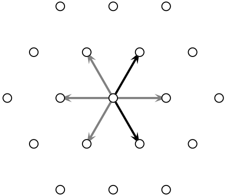
- *Fundamental roots*
- A finite reflection group can be re-formulated by a *root system* and studied via linear algebra.

Root Lattices

- Not all root systems *generate* lattices!
  (Should be shift-invariant.)

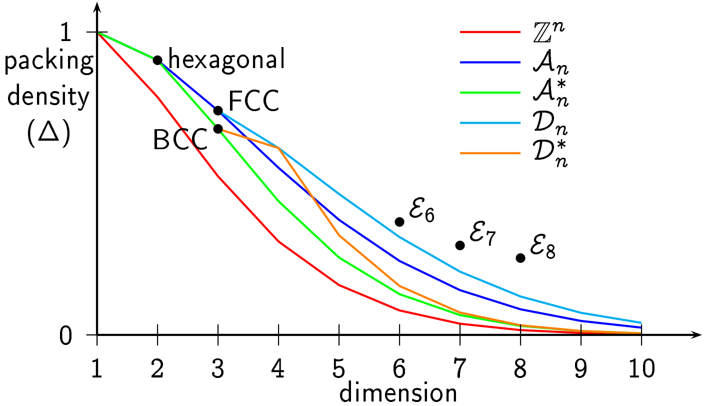


- *Crystallographic restriction*: Dihedral angles are limited to π/k, $k \in \{2, 3, 4, 6\}$.



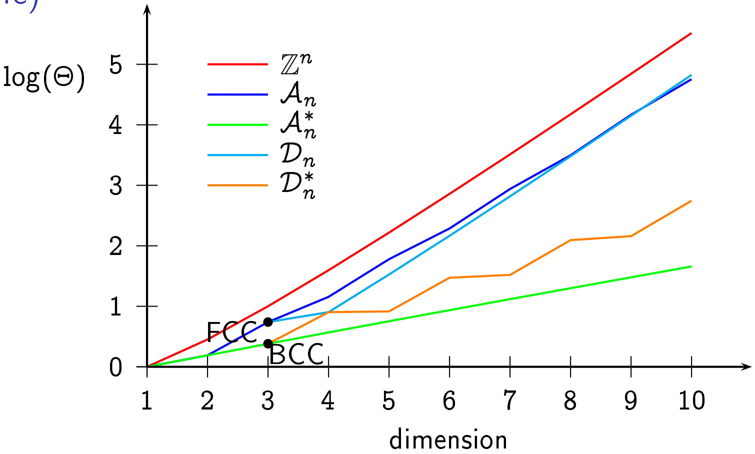
- Symmetric: Root lattices have the same symmetry as the finite reflection group at every lattice point.

Packing Densities of Some Root Lattices (Conway & Sloane)



- Known to be optimal (among lattices) up to dimension 8: $\mathbb{Z} \simeq \mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3 \simeq \mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_5$, $\mathcal{E}_6$, $\mathcal{E}_7$, and $\mathcal{E}_8$
- Cartesian lattices are not efficient sampling lattices.

Covering Thickness (Θ) of Some Root Lattices (Conway & Sloane)



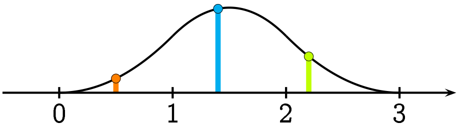
- Known to be optimal (among lattices) up to dimension 5: $\mathbb{Z}$, $\mathcal{A}_2$, $\mathcal{A}_3^*$, $\mathcal{A}_4^*$, and $\mathcal{A}_5^*$
- Again, Cartesian lattices are not efficient sampling lattices.

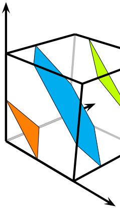
- Root lattices are good candidates for efficient sampling in arbitrary dimensions.
- Cartesian lattices are less efficient sampling than other root lattices.
- Which (symmetric) reconstruction filter can we use?
  $\rightarrow$ **Box-splines**

Box-Splines: Definition

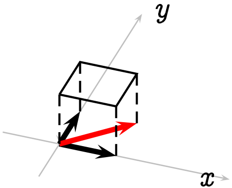
$n \times m$ Direction matrix

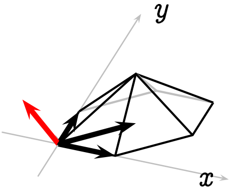
$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

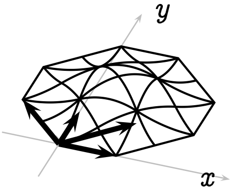




$$\begin{bmatrix} 1 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$







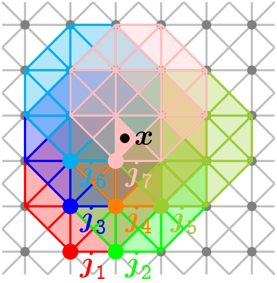
Box-Splines: Properties

- Finite support defined by Minkowski sum of the directions.
- Piecewise polynomial of degree $m - n$
- Polynomial pieces are delineated by the shifts of the *knot planes* (Hyperplanes spanned by the directions of Ξ).
- Carl De Boor, Klaus Höllig, S. D. Riemenschneider "Box Splines" (1993)

Spline

- A linear combination of the shifts of the box-spline: $s \in S_{\Xi} := \mathrm{span}(M_{\Xi}(\cdot - \boldsymbol{j}))_{\boldsymbol{j} \in \mathbb{Z}^n}$.
- $\{M_{\Xi}(\cdot - \boldsymbol{j})\}_{\boldsymbol{j} \in \mathbb{Z}^n}$ form a basis iff $\boldsymbol{\Xi}$ is unimodular.
- Approximation order (when $\boldsymbol{\Xi} \in \mathbb{Z}^{n \times m}$) $\rho(\boldsymbol{\Xi}) := \{\min_{\mathbf{Z} \subseteq \boldsymbol{\Xi}} \#\mathbf{Z} : \mathrm{rank}(\boldsymbol{\Xi} \backslash \mathbf{Z}) < n\}$
- Spline evaluation at $\boldsymbol{x}$.

$$
\begin{aligned}
s(\boldsymbol{x}) = \sum_{\boldsymbol{j} \in \mathbb{Z}^n} & M_{\Xi}(\boldsymbol{x} - \boldsymbol{j}) a(\boldsymbol{j}) \\
= & M_{\Xi}(\boldsymbol{x} - \boldsymbol{j}_1) a(\boldsymbol{j}_1) \\
+ & M_{\Xi}(\boldsymbol{x} - \boldsymbol{j}_2) a(\boldsymbol{j}_2) \\
+ & M_{\Xi}(\boldsymbol{x} - \boldsymbol{j}_3) a(\boldsymbol{j}_3) \\
+ & M_{\Xi}(\boldsymbol{x} - \boldsymbol{j}_4) a(\boldsymbol{j}_4) \\
+ & M_{\Xi}(\boldsymbol{x} - \boldsymbol{j}_5) a(\boldsymbol{j}_5) \\
+ & M_{\Xi}(\boldsymbol{x} - \boldsymbol{j}_6) a(\boldsymbol{j}_6) \\
+ & M_{\Xi}(\boldsymbol{x} - \boldsymbol{j}_7) a(\boldsymbol{j}_7)
\end{aligned}
$$



- Large support $\to$ More samples for evaluation

Box-Splines on Non-Cartesian Lattices

$$\sum_{\boldsymbol{j} \in \mathbf{L}\mathbb{Z}^n} |\det \mathbf{L}| M_{\mathbf{L}\boldsymbol{\Xi}}(\cdot - \boldsymbol{j}) a(\boldsymbol{j}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^n} M_{\boldsymbol{\Xi}}(\mathbf{L}^{-1} \cdot - \boldsymbol{k}) a(\mathbf{L}\boldsymbol{k})$$

- A spline as a linear combination of the shifts of the box-spline $|\det \mathbf{L}| M_{\mathbf{L}\boldsymbol{\Xi}}$ on the non-Cartesian lattice $\mathbf{L}\mathbb{Z}^n$ has a *change of variables* relation with the spline as a linear combination of the shifts of the box-spline $M_{\boldsymbol{\Xi}}$ on the Cartesian lattice.

Constructing Symmetric Box-Splines on Root Lattices

- To maximize the approximation order, directions associated with the lattice points are used. This also guarantees rational polynomial coefficients. (Kim & Peters '09)
- To make the box-spline has the same symmetry as the lattice, all the (non-parallel) lattice points with the same distances are included.
- To make the support of the box-spline small, consider the short directions first.

Symmetric Box-Spline on the Cartesian Lattice

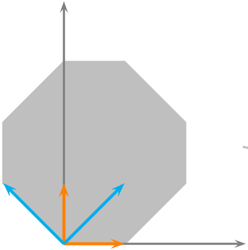
dim.	box-spline	lattice	direction matrix	generator matrix	continuity	basis?	$\lambda_*(f(\cdot + j))$
n	$M_{\mathbb{Z}^n}$	Cartesian	$\mathbf{I}_n \cup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j\}$	$\mathbf{I}_n$	$C^{2^{n-2}}$	no	not known
2	$M_{\mathbb{Z}^2} \cong$ ZP-element		$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		C^1	no	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{\mathrm{ZP}}} D_\xi^2 f)(j)$
3	$M_{\mathbb{Z}^3}$		$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		C^2	no	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{\mathbb{Z}^3}} D_\xi^2 f)(j)$
n	$M^{\pm}_{\mathcal{A}_n}$	$\mathcal{A}_n$	$\bigcup_{1 \le i < j \le n+1} \{\mathbf{X}_n^{\pm}(\mathbf{e}_i - \mathbf{e}_j)\}$	$\mathbf{A}_n^{\pm}$	C^{n-2}	yes	not known
2	$M^{\pm}_{\mathcal{A}_2} \cong M^{\pm}_{\mathcal{A}_2^*} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M^{-}_{\mathcal{A}_3} = M_{\mathrm{fcc}} \cong M_{\mathcal{D}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M^{\pm}_{\mathcal{A}_n^*} = M_1^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^0	yes	$f(j)$
n	$M_r^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm} \bigcup_{i=1}^{r} \begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^{2r-2}	yes	not known
n	$M_2^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} & \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^2	yes	$(f - \frac{1}{12} \sum_{\xi \in \mathbf{T}_2^{*\pm}} D_\xi^2 f)(j)$
2	$M^{\pm}_{\mathcal{A}_2^*} \cong M^{\pm}_{\mathcal{A}_2} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2 \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M^{\pm}_{\mathcal{A}_3^*} \cong M^{\pm}_{\mathcal{A}_3} \cong M_1^{*\pm}$	BCC	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	C^0	yes	$f(j)$
n	$M_{\mathcal{D}_n}$	$\mathcal{D}_n$	$\bigcup_{1 \le i < j \le n} \{\mathbf{e}_i \pm \mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & -\mathbf{e}_{n-1} \\ -\mathbf{j}^t & -1 \end{bmatrix}$	$\begin{cases} C^1 & (n = 3) \\ C^{2n-4} & (n > 3) \end{cases}$	$\begin{cases} \text{yes} & (n = 3) \\ \text{no} & (n > 3) \end{cases}$	not known
3	$M_{\mathcal{D}_3} \cong M^{\pm}_{\mathcal{A}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{\mathrm{fcc}}} D_\xi^2 f)(j)$
n	$M_{\mathcal{D}_n^*}$	$\mathcal{D}_n^*$	$\mathbf{I}_n \cup \frac{1}{2}\{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{j}/2 \\ \mathbf{0}^t & 1/2 \end{bmatrix}$	$C^{2^{n-2}}$	no	not known
3	$M_{\mathcal{D}_3^*}$	BCC	$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$	C^2	no	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{\mathrm{bcc}}} D_\xi^2 f)(j)$

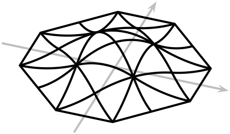
Symmetric Box-Spline on the Cartesian Lattice

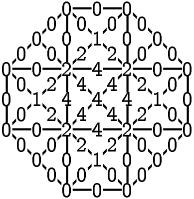
- The Cartesian lattice $\mathbb{Z}^n$
  - Generated by the root system
    $\mathscr{B}_n := \{\pm\mathbf{e}_i \pm \mathbf{e}_j : 1 \leq i \neq j \leq n\} \cup \bigcup_{1\leq j\leq n}\{\mathbf{e}_j\}$
  - Symmetry order: $2^n n!$
  - Center density: 2^{-n}
- (Symmetric) tensor-product B-spline
  - Constructed by the n shortest (axis-aligned) directions, repeated r times each.
  - High degree (rn) compared to its approximation order $(r-1)$.
  - Too large support $\rightarrow$ High computationl cost
- The symmetric box-spline $M_{\mathbb{Z}^n}$
  - Constructed by n axis-aligned directions + 2^{n-1} diagonal directions
    $\rightarrow$ Extension of ZP-element (Zwart '73) and 7-direction box-spline (Peters '96)
  - Polynomial degree: 2^{n-1}
  - Approximation order: $2^{n-2}+2$
  - Shifts do not form a basis.

Box-Spline $M_{\mathbb{Z}^2}$ on the Cartesian Lattice

- Direction matrix $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.
- Centered ZP-element (Zwart '73).
- Piecewise polynomial of degree 2.
- C^1 continuous & approximation order 3
- Stencil size is 7.

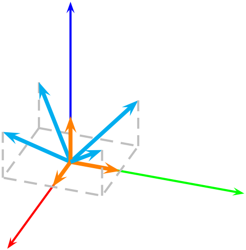


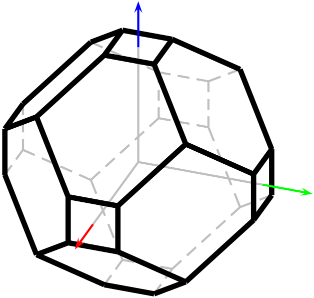




Box-Spline $M_{\mathbb{Z}^3}$ on the Cartesian Lattice

- Direction matrix $\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.
- Centered 7-direction box-spline (Peters '96).
- Piecewise polynomial of degree 4.
- C^2 continuous & approximation order 4
- Stencil size is 53.
  cf. 64 for B-spline with the same approximation order





Symmetric Box-Spline on the $\mathcal{A}_n$ Lattice

dim.	box-spline	lattice	direction matrix	generator matrix	continuity	basis?	$\lambda_*(f(\cdot + j))$
n	$M_{\mathbb{Z}^n}$	Cartesian	$\mathbf{I}_n \cup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm\mathbf{e}_j\}$	$\mathbf{I}_n$	$C^{2^{n-2}}$	no	not known
2	$M_{\mathbb{Z}^2} \cong$ZP-element		$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		C^1	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{\mathrm{ZP}}} D_\xi^2 f)(j)$
3	$M_{\mathbb{Z}^3}$		$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		C^2	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{\mathbb{Z}^3}} D_\xi^2 f)(j)$
n	$M^{\pm}_{\mathcal{A}_n}$	$\mathcal{A}_n$	$\bigcup_{1 \le i < j \le n+1} \{\mathbf{X}_n^{\pm}(\mathbf{e}_i - \mathbf{e}_j)\}$	$\mathbf{A}_n^{\pm}$	C^{n-2}	yes	not known
2	$M^{\pm}_{\mathcal{A}_2} \cong M^{\pm}_{\mathcal{A}_2^*} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M^{-}_{\mathcal{A}_3} = M_{\mathrm{fcc}} \cong M_{\mathcal{D}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M^{\pm}_{\mathcal{A}_n^*} = M_1^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^0	yes	$f(j)$
n	$M_r^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\bigcup_{i=1}^{r}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^{2r-2}	yes	not known
n	$M_2^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} & \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^2	yes	$(f - \frac{1}{12}\sum_{\xi \in \mathbf{T}_2^{*\pm}} D_\xi^2 f)(j)$
2	$M^{\pm}_{\mathcal{A}_2^*} \cong M^{\pm}_{\mathcal{A}_2} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2 \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M^{\pm}_{\mathcal{A}_3^*} \cong M^{\pm}_{\mathcal{A}_3} \cong M_1^{*\pm}$	BCC	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	C^0	yes	$f(j)$
n	$M_{\mathcal{D}_n}$	$\mathcal{D}_n$	$\bigcup_{1 \le i < j \le n} \{\mathbf{e}_i \pm \mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & -\mathbf{e}_{n-1} \\ -\mathbf{j}^t & -1 \end{bmatrix}$	$\begin{cases} C^1 & (n = 3) \\ C^{2n-4} & (n > 3) \end{cases}$	$\begin{cases} \text{yes} & (n = 3) \\ \text{no} & (n > 3) \end{cases}$	not known
3	$M_{\mathcal{D}_3} \cong M^{\pm}_{\mathcal{A}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M_{\mathcal{D}_n^*}$	$\mathcal{D}_n^*$	$\mathbf{I}_n \cup \frac{1}{2}\{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm\mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{j}/2 \\ \mathbf{0}^t & 1/2 \end{bmatrix}$	$C^{2^{n-2}}$	no	not known
3	$M_{\mathcal{D}_3^*}$	BCC	$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$	C^2	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{\mathrm{bcc}}} D_\xi^2 f)(j)$

Symmetric Box-Spline on the $\mathcal{A}_n$ Lattice

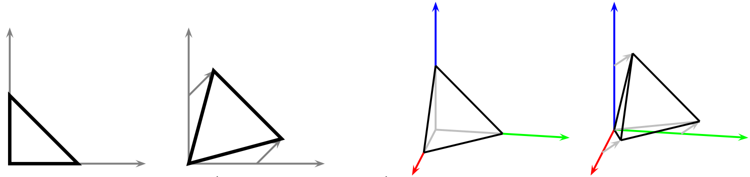
- The $\mathcal{A}_n$ lattice
  - Generated by the root system $\mathscr{A}_n := \{\pm(\mathbf{e}_i - \mathbf{e}_j) \in \mathbb{R}^{n+1} : 1 \leq i \neq j \leq n+1\}$
  - Generated by the vectors associated with the n edges of a regular n-simplex sharing a vertex.
  - Symmetry order: $(n+1)!2$
  - Center density: $2^{n/2}(n+1)^{-1/2}$
  - Examples: hexagonal, FCC
- The symmetric box-spline $M^{\pm}_{\mathcal{A}_n}$
  - Constructed by the shortest (non-parallel) $n(n+1)/2$ directions.
  - Polynomial degree: $n(n-1)/2$
  - Approximation order: n
  - The shifts form a basis.

Embedding the $\mathcal{A}_n$ Lattice in $\mathbb{R}^n$

- Diagonally scale the n-simplex composed of

$$\{\mathbf{0}\} \cup \bigcup_{1 \le j \le n} \{\mathbf{e}_j\}$$

such that it becomes regular.

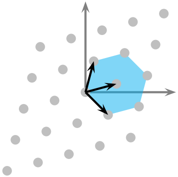


$\rightarrow$ $\mathbf{A}_n^{\pm} := \mathbf{I}_n + \frac{1}{n}\left(-1 \pm \sqrt{n+1}\right)\mathbf{J}_n$

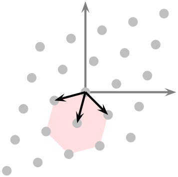
$\rightarrow$ Square generator matrices of the $\mathcal{A}_n$ lattice (Kim & Peters '10)

Box-Spline $M^{\pm}_{\mathcal{A}_2}$ on the Hexagonal Lattice

- Generator matrix $\frac{1}{2}\begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$.
- Direction matrix $\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$.
- Bivariate linear box-spline on the hexagonal lattice.



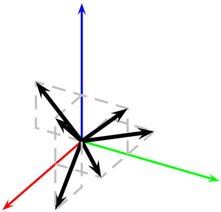
$M^{+}_{\mathcal{A}_2}$

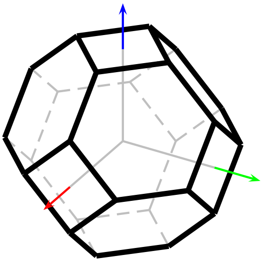


$M^{-}_{\mathcal{A}_2}$

Box-Spline M_{fcc} on the FCC Lattice

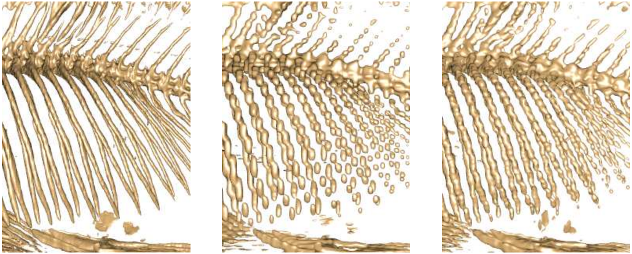
- Generator matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
- Direction matrix $\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$.
- Six-direction box-spline on the FCC lattice (Entezari '07).





Reconstruction on the FCC Lattice

- Quality



- Performance

Dataset	Cartesian	FCC	Ratio
Marschner-Lobb	135	98	72%
Carp	515	358	69%

Minho Kim, Alireza Entezari and Jörg Peters, *Box-Spline Reconstruction on the Face Centered Cubic lattice*, IEEE Visualization 2008.

Symmetric Box-Spline on the $\mathcal{A}_n^*$ Lattice

dim.	box-spline	lattice	direction matrix	generator matrix	continuity	basis?	$\lambda_*(f(\cdot + j))$
n	$M_{\mathbb{Z}^n}$	Cartesian	$\mathbf{I}_n \cup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm\mathbf{e}_j\}$	$\mathbf{I}_n$	$C^{2^{n-2}}$	no	not known
2	$M_{\mathbb{Z}^2} \cong$ZP-element		$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		C^1	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{ZP}} D_\xi^2 f)(j)$
3	$M_{\mathbb{Z}^3}$		$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		C^2	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{\mathbb{Z}^3}} D_\xi^2 f)(j)$
n	$M^{\pm}_{\mathcal{A}_n}$	$\mathcal{A}_n$	$\bigcup_{1 \le i < j \le n+1} \{\mathbf{X}_n^{\pm}(\mathbf{e}_i - \mathbf{e}_j)\}$	$\mathbf{A}_n^{\pm}$	C^{n-2}	yes	not known
2	$M^{\pm}_{\mathcal{A}_2} \cong M^{\pm}_{\mathcal{A}_2^*} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M^{-}_{\mathcal{A}_3} = M_{\text{fcc}} \cong M_{\mathcal{D}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M^{\pm}_{\mathcal{A}_n^*} = M_1^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^0	yes	$f(j)$
n	$M_r^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\bigcup_{i=1}^{r}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^{2r-2}	yes	not known
n	$M_2^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} & \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^2	yes	$(f - \frac{1}{12}\sum_{\xi \in \mathbf{T}_2^{*\pm}} D_\xi^2 f)(j)$
2	$M^{\pm}_{\mathcal{A}_2^*} \cong M^{\pm}_{\mathcal{A}_2} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2 \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M^{\pm}_{\mathcal{A}_3^*} \cong M^{\pm}_{\mathcal{A}_3} \cong M_1^{*\pm}$	BCC	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	C^0	yes	$f(j)$
n	$M_{\mathcal{D}_n}$	$\mathcal{D}_n$	$\bigcup_{1 \le i < j \le n} \{\mathbf{e}_i \pm \mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & -\mathbf{e}_{n-1} \\ -\mathbf{j}^t & -1 \end{bmatrix}$	$\begin{cases} C^1 & (n=3) \\ C^{2n-4} & (n>3) \end{cases}$	$\begin{cases} \text{yes} & (n=3) \\ \text{no} & (n>3) \end{cases}$	not known
3	$M_{\mathcal{D}_3} \cong M^{\pm}_{\mathcal{A}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M_{\mathcal{D}_n^*}$	$\mathcal{D}_n^*$	$\mathbf{I}_n \cup \frac{1}{2}\{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm\mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{j}/2 \\ \mathbf{0}^t & 1/2 \end{bmatrix}$	$C^{2^{n-2}}$	no	not known
3	$M_{\mathcal{D}_3^*}$	BCC	$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$	C^2	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{bcc}} D_\xi^2 f)(j)$

Symmetric Box-Spline on the $\mathcal{A}_n^*$ Lattice

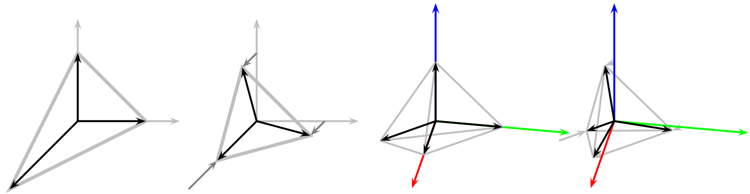
- The $\mathcal{A}_n^*$ lattice
  - The dual lattice of the $\mathcal{A}_n$ lattice
  - Generated by the vectors from the center of a regular n-simplex to its vertices.
  - Symmetry order: $(n+1)!2$
  - Center density: $\frac{n^{n/2}}{2^n(n+1)^{(n-1)/2}}$
  - Examples: hexagonal, BCC
- The symmetric box-spline $M^{\pm}_{\mathcal{A}_n^*}$
  - Constructed by the $(n+1)$ shortest directions.
  - Polynomial degree: 1
  - Approximation order: 2
  - The shifts form a basis.
  - Examples: 4- and 8-direction box-splines on the BCC lattice (Entezari et al.)

Embedding the $\mathcal{A}_n^*$ Lattice in $\mathbb{R}^n$

- Diagonally scale the n-simplex composed of

$$\{-\mathbf{j}\} \cup \bigcup_{1 \leq j \leq n} \{\mathbf{e}_j\}$$

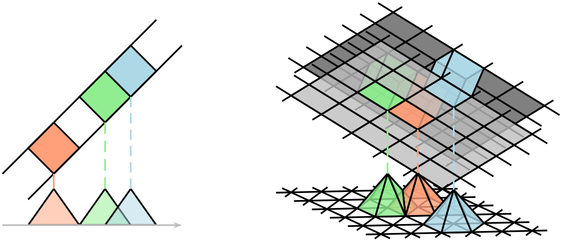
such that it becomes regular.



$\rightarrow \mathbf{A}_n^{*\pm} := \mathbf{I}_n + \frac{1}{n}\left(-1 \pm \frac{1}{\sqrt{n+1}}\right)\mathbf{J}_n$
$\rightarrow$ Square generator matrices of the $\mathcal{A}_n^*$ lattice
(Kim & Peters '10)

Symmetric Linear Box-Spline on the $\mathcal{A}_n^*$ Lattice

- Analogous to the shadow projection of a slab along diagonal.



Minho Kim, Jörg Peters, *Symmetric Box-Splines on the $\mathcal{A}_n^*$ Lattice*
Journal of Approximation Theory 2010.

Symmetric Box-Spline on the $\mathcal{D}_n$ Lattice

dim.	box-spline	lattice	direction matrix	generator matrix	continuity	basis?	$\lambda_*(f(\cdot + j))$
n	$M_{\mathbb{Z}^n}$	Cartesian	$\mathbf{I}_n \cup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm\mathbf{e}_j\}$	$\mathbf{I}_n$	$C^{2^{n-2}}$	no	not known
2	$M_{\mathbb{Z}^2}$ ≅ZP-element		$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		C^1	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{ZP}} D_\xi^2 f)(j)$
3	$M_{\mathbb{Z}^3}$		$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		C^2	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{\mathbb{Z}^3}} D_\xi^2 f)(j)$
n	$M_{\mathcal{A}_n}^{\pm}$	$\mathcal{A}_n$	$\bigcup_{1 \le i < j \le n+1} \{\mathbf{X}_n^{\pm}(\mathbf{e}_i - \mathbf{e}_j)\}$	$\mathbf{A}_n^{\pm}$	C^{n-2}	yes	not known
2	$M_{\mathcal{A}_2}^{\pm} \cong M_{\mathcal{A}_2^*}^{\pm} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M_{\mathcal{A}_3}^{-} = M_{\text{fcc}} \cong M_{\mathcal{D}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M_{\mathcal{A}_n^*}^{\pm} = M_1^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^0	yes	$f(j)$
n	$M_r^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\bigcup_{i=1}^{r}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^{2r-2}	yes	not known
n	$M_2^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} & \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^2	yes	$(f - \frac{1}{12}\sum_{\xi \in \mathbf{T}_2^{*\pm}} D_\xi^2 f)(j)$
2	$M_{\mathcal{A}_2^*}^{\pm} \cong M_{\mathcal{A}_2}^{\pm} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2 \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M_{\mathcal{A}_3^*}^{\pm} \cong M_{\mathcal{A}_3}^{\pm} \cong M_1^{*\pm}$	BCC	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	C^0	yes	$f(j)$
n	$M_{\mathcal{D}_n}$	$\mathcal{D}_n$	$\bigcup_{1 \le i < j \le n} \{\mathbf{e}_i \pm \mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & -\mathbf{e}_{n-1} \\ -\mathbf{j}^t & -1 \end{bmatrix}$	$\begin{cases} C^1 & (n = 3) \\ C^{2n-4} & (n > 3) \end{cases}$	$\begin{cases} \text{yes} & (n = 3) \\ \text{no} & (n > 3) \end{cases}$	not known
3	$M_{\mathcal{D}_3} \cong M_{\mathcal{A}_3}^{\pm}$	FCC	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M_{\mathcal{D}_n^*}$	$\mathcal{D}_n^*$	$\mathbf{I}_n \cup \frac{1}{2}\{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm\mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{j}/2 \\ \mathbf{0}^t & 1/2 \end{bmatrix}$	$C^{2^{n-2}}$	no	not known
3	$M_{\mathcal{D}_3^*}$	BCC	$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$	C^2	no	$(f - \frac{1}{24}\sum_{\xi \in \Xi_{bcc}} D_\xi^2 f)(j)$

Symmetric Box-Spline on the $\mathcal{D}_n$ Lattice

- The $\mathcal{D}_n$ lattice
  - A.k.a. "checkerboard lattice" $\{\boldsymbol{j} \in \mathbb{Z}^n : \sum_k \boldsymbol{j}(k) \text{ is even}\}$
  - Generated by the root system $\mathscr{C}_n$ or $\mathscr{D}_n$
  - Defined only for $n \geq 3$
  - Symmetry order: $\begin{cases} 2^n n! & (n \neq 4) \\ 1152 & (n = 4) \end{cases}$
  - Center density: $2^{(n+2)/2}$
  - Example: FCC
- The symmetric box-spline $M_{\mathcal{D}_n}$
  - Constructed by the $n(n-1)$ shortest directions
  - Polynomial degree: $n(n-2)$
  - Approximation order: $2n-2$
  - The shifts do not form a basis except for $n = 3$.
  - Example: 6-direction box-splines on the FCC lattice

Symmetric Box-Spline on the $\mathcal{D}_n^*$ Lattice

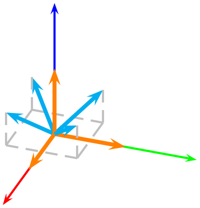
dim.	box-spline	lattice	direction matrix	generator matrix	continuity	basis?	$\lambda_*(f(\cdot + j))$
n	$M_{\mathbb{Z}^n}$	Cartesian	$\mathbf{I}_n \cup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j\}$	$\mathbf{I}_n$	$C^{2^{n-2}}$	no	not known
2	$M_{\mathbb{Z}^2} \cong$ZP-element		$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		C^1	no	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{ZP}} D_\xi^2 f)(j)$
3	$M_{\mathbb{Z}^3}$		$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		C^2	no	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{\mathbb{Z}^3}} D_\xi^2 f)(j)$
n	$M^{\pm}_{\mathcal{A}_n}$	$\mathcal{A}_n$	$\bigcup_{1 \le i < j \le n+1} \{\mathbf{X}_n^{\pm}(\mathbf{e}_i - \mathbf{e}_j)\}$	$\mathbf{A}_n^{\pm}$	C^{n-2}	yes	not known
2	$M^{\pm}_{\mathcal{A}_2} \cong M^{\pm}_{\mathcal{A}_2^*} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M^{-}_{\mathcal{A}_3} = M_{\text{fcc}} \cong M_{\mathcal{D}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M^{\pm}_{\mathcal{A}_n^*} = M_1^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^0	yes	$f(j)$
n	$M_r^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm} \bigcup_{i=1}^{r} \begin{bmatrix} \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^{2r-2}	yes	not known
n	$M_2^{*\pm}$	$\mathcal{A}_n^*$	$\mathbf{A}_n^{*\pm}\begin{bmatrix} \mathbf{I}_n & -\mathbf{j} & \mathbf{I}_n & -\mathbf{j} \end{bmatrix}$	$\mathbf{A}_n^{*\pm}$	C^2	yes	$(f - \frac{1}{12} \sum_{\xi \in \mathbf{T}_2^{*\pm}} D_\xi^2 f)(j)$
2	$M^{\pm}_{\mathcal{A}_2^*} \cong M^{\pm}_{\mathcal{A}_2} \cong M_1^{*\pm}$	hexagonal	$\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2 \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^0	yes	$f(j)$
3	$M^{\pm}_{\mathcal{A}_3^*} \cong M^{\pm}_{\mathcal{A}_3} \cong M_1^{*\pm}$	BCC	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	C^0	yes	$f(j)$
n	$M_{\mathcal{D}_n}$	$\mathcal{D}_n$	$\bigcup_{1 \le i < j \le n} \{\mathbf{e}_i \pm \mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & -\mathbf{e}_{n-1} \\ -\mathbf{j}^t & -1 \end{bmatrix}$	$\begin{cases} C^1 & (n = 3) \\ C^{2n-4} & (n > 3) \end{cases}$	$\begin{cases} \text{yes} & (n = 3) \\ \text{no} & (n > 3) \end{cases}$	not known
3	$M_{\mathcal{D}_3} \cong M^{\pm}_{\mathcal{A}_3}$	FCC	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	C^1	yes	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{fcc}} D_\xi^2 f)(j)$
n	$M_{\mathcal{D}_n^*}$	$\mathcal{D}_n^*$	$\mathbf{I}_n \cup \frac{1}{2}\{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j\}$	$\begin{bmatrix} \mathbf{I}_{n-1} & \mathbf{j}/2 \\ \mathbf{0}^t & 1/2 \end{bmatrix}$	$C^{2^{n-2}}$	no	not known
3	$M_{\mathcal{D}_3^*}$	BCC	$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$	C^2	no	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{bcc}} D_\xi^2 f)(j)$

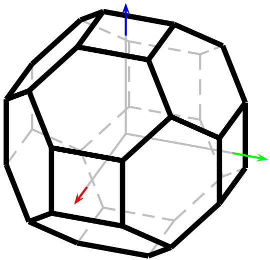
Symmetric Box-Spline on the $\mathcal{D}_n^*$ Lattice

- The $\mathcal{D}_n^*$ lattice
  - The dual lattice of the $\mathcal{D}_n$ lattice
  - Generated by inserting additional points at the center of the cubes embedded in $\mathbb{Z}^n$. $\rightarrow$ "Body-centered cubic lattice"
  - Symmetric order: same as that of the $\mathcal{D}_n$ lattice
  - Center density: $\begin{cases} 3^{1.5}2^{-5} & (n=3) \\ 2^{-(n-1)} & (n>3) \end{cases}$
  - Example: BCC
- The symmetric box-spline $M_{\mathcal{D}_n^*}$
  - Polynomial degree: 2^{n-1}
  - Approximation order: $2^{n-2}+2$
  - The shifts do not form a basis.

Box-Spline $M_{\mathcal{D}_3^*}$ on the BCC Lattice

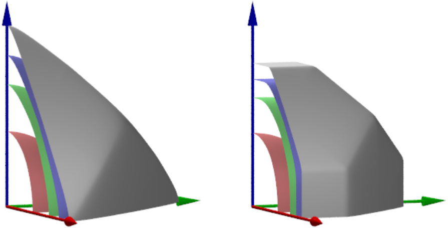
- Direction matrix $\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$.
- Generator matrix $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.



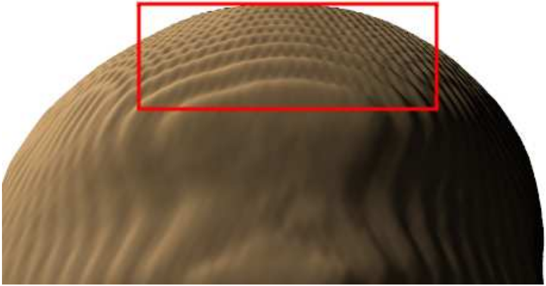


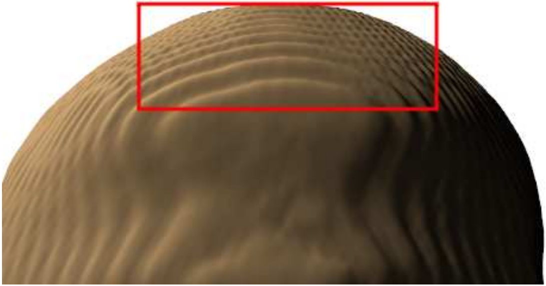
$M_{\mathcal{D}_3^*}$ vs. 8-Direction Box-Spline (Entezari *et al.* '04)

box-spline	8-dir.	$M_{\mathcal{D}_3^*}$
polynomial degree	5	4
approximation order	3	3
# of pieces	192	720
stencil size	32	30
basis?	yes	no



$M_{\mathcal{D}_3^*}$ vs. 8-Direction Box-Spline (Entezari *et al.* '04)





Wrap-Up

- Root lattices ⇒ Good!
- Box-splines ⇒ Good!
- Box-splines on root lattices ⇒ **Awesome!!!**

WANTED: Applications in high dimensions

Questions?